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# Invariant Subspaces Containing all Constant Directions*

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A closed subspace $\mathscr{M}$ of $H^2_{\mathscr{H}}$ invariant under the shift operator which contains for each $e \in \mathscr{H}$ a function of the form $qe$, where $q$ is inner and depends on $e$, contains a subspace of the form $qH^2_{\mathscr{H}}$; that is $q$ can be chosen independently of $e$. This theorem is generalized and proved and its relationship to the local characterization of $C_0$ operators of Sz.-Nagy and Foiaş (*J. Func. Anal.* 8 (1961), 76) is discussed.

Let $H^2_{\mathscr{H}}$ denote the (separable) Hilbert space of all functions defined on the unit circle, taking values in the separable Hilbert space $\mathscr{H}$, and which are weakly in the Hardy class $H^2$. A closed subspace $\mathscr{M} \subset H^2_{\mathscr{H}}$ is said to be invariant if it is invariant under the right-shift operator $S$ defined by $(SF)(e^{i\theta}) = e^{i\theta}F(e^{i\theta})$. Let $F \in H^2_{\mathscr{H}}$. An invariant subspace $\mathscr{M}$ contains the direction of $F$ if there exists a scalar function $f$ such that $fF \in \mathscr{M}$. In this case there is an inner function $q$, and in fact, a minimal inner function $q_F$ such that $q_F F \in \mathscr{M}$ [5, p. 165]. An invariant subspace contains all (analytic) directions if it contains the direction of $F$ for every $F \in H^2_{\mathscr{H}}$. An invariant subspace $\mathscr{M}$ contains all constant directions if for every $e \in \mathscr{H}$ (thought of as a constant function in $H^2_{\mathscr{H}}$) there exists a scalar function $f_e$ (and therefore a minimal inner function $q_e$), such that $f_e e \in \mathscr{M}$ (and $q_e e \in \mathscr{M}$). In Ref. [5], the author conjectured the following:

THEOREM 1. *If an invariant subspace contains all analytic directions, there exists a scalar inner function $q$ such that $\mathscr{M} \supset qH^2_{\mathscr{H}}$.*

Theorem 1 was established by Sz.-Nagy and Foiaş [4], where they, in fact, proved an operator theoretic generalization. The purpose of this paper is to generalize Theorem 1 in another direction as follows:

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THEOREM 2. *Let $K$ be a closed subspace of $H^2_{\mathscr{H}}$ such that $v_0^\infty S^n K = H^2_{\mathscr{H}}$. ($v_0^\infty S^n K$ denotes the closed subspace generated by $K$, $SK$, $S^2K$,...). If an invariant subspace $\mathscr{M}$ contains all directions corresponding to elements of $K$, then there exists a scalar inner function $q$ such that $\mathscr{M} \supset qH^2_{\mathscr{H}}$.*

The most interesting special case of Theorem 2 is when $K$ consists of the constant functions $\mathscr{C}$. Theorem 2 generalizes the authors results [5] and a theorem of D. Herrero [2], neither of which are implied by the main result of Ref. [4], where, where, in effect, all directions, not just constant directions are assumed. Sz.-Nagy and Foiaş have pointed out to the author an operator theoretic generalization of Theorem 2, which is stated below as Theorem 3.

Let $\mathscr{M} \subset H^2_{\mathscr{H}}$ be invariant and let $H = H^2_{\mathscr{H}} \ominus \mathscr{M}$. Let $T = PS \mid H$, where $P$ is the orthogonal projection of $H^2_{\mathscr{H}}$ onto $H$ and $S$ is the shift operator defined above. Then $(T^*F)(e^{i\theta}) = e^{-i\theta}[F(e^{i\theta}) - F(0)]$, so $T^{*n} \to 0$ strongly and $T$ is completely nonunitary. Conversely, given a contraction $R$ such that $R^{*n} \to 0$ strongly, $R$ is unitarily equivalent to an operator arising from the above construction [1]. The functional calculus $\varphi \to \varphi(T)$ of Ref. [3, p. 101] reduces here to $\varphi(T) = PM_\varphi \mid H$, where $\varphi \in H^\infty$ and $M_\varphi$ is multiplication by $\varphi$. For $F \in H^2_{\mathscr{H}}$, $qF \in \mathscr{M} \Leftrightarrow P(qF) = 0 \Leftrightarrow P(qPF) = 0 \Leftrightarrow q(T)(PF) = 0$. Thus $\mathscr{M}$ contains the direction of $F$ if and only if the restriction of $T$ to the closed subspace of $H$ generated by $\{PF, TPF, T^2PF,...\}$ belongs to $C_0$ on that subspace. ($C_0$ is the class of completely nonunitary contractions annihilated by an $H^\infty$ function [3, p. 113].) Thus Theorem 1 is equivalent to the following: Let $T$ be a contraction such that $T^{*n} \to 0$ strongly. Then if the restriction of $T$ to every cyclic subspace belongs to $C_0$ on that subspace, $T$ itself belongs to $C_0$. Sz.-Nagy and Foiaş proved Theorem 1 in this setting [4], but without assuming $T^{*n} \to 0$. Of course, the theorem once proved, implies $T^{*n} \to 0$, since this is true generally for operators in $C_0$ [3, p. 114].

We begin the proof of Theorem 2 with a strengthened form of Lemma 5 of Ref. [5].

LEMMA 1. *Let $F_1$ and $F_2$ be linearly independent in $H^2_{\mathscr{H}}$ and let $\mathscr{M}$ be an invariant subspace which contains the directions of $F_1$ and $F_2$. Let $H_0$ be the two-dimensional subspace of $H^2_{\mathscr{H}}$ generated by $F_1$, $F_2$. Then for all but countably many unit vectors $G \in H_0$, $q_G = q_{F_1} \wedge q_{F_2}$.*

*Proof.* Let $\mathscr{N}$ be the closed invariant span of $F_1$ and $F_2$ and let

$q$ be the minimal inner function such that $\mathcal{M} \supset q\mathcal{N}$. Then the minimal inner functions $q_1 = q_{F_1}$ and $q_2 = q_{F_2}$ are of the form $q_1 = \bar{p}_1 q$ and $q_2 = \bar{p}_2 q$, where $p_1$, $p_2$ are inner. Clearly $q = q_1 \wedge q_2 = q/p_1 \wedge q/p_2 = q/(p_1 \vee p_2)$, where as in Ref. [5, p. 165] we set $(q_1 \wedge q_2) H^2 = q_1 H^2 \cap q_2 H^2$ and $(q_1 \wedge q_2) H^2 =$ invariant span of $q_1 H^2$ and $q_2 H^2$. Therefore, $p_1 \vee p_2 = 1$, and if $H_0$ (and therefore $\mathcal{N}$) are fixed, this is true for any linearly independent choice of $F_1$ and $F_2$ in $H_0$. Thus, the inner factors $p$ of $q$ such that $\bar{p}q$ is the minimal inner function of some $F \in H_0$ have the property that no two $p$'s corresponding to linearly independent $F$'s have any common inner factor. Since the zeros of $q$ are a countable set, and the singular measure of $q$ can dominate only countably many mutually singular measures, the number of nonconstant $p$'s which can occur is countable, from which the result follows:

*Proof of Theorem* 2. Let $\{\lambda_n\}$ be a sequence of points in the open unit disk such that $\lambda_n \to 0$. Then proceeding in a manner analogous to Ref. [4], we define $\Gamma_{ij} = \{F \in K : | q_F(\lambda_i)| \geqslant 1/j\}$. Then $\bigcup_{i,j} \Gamma_{ij} = K$, since if $F \notin \bigcup \Gamma_{ij}$, then $| q_F(\lambda_i)| = 0$ for all $i$ and therefore $q_F = 0$. (We set $q_0 = 1$, so that the zero vector belongs to every $\Gamma_{ij}$.) We need to prove the $\Gamma_{ij}$ are closed. Let $F_n \to F$, where every $F_n \in \Gamma_{ij}$. By passing to a subsequence, if necessary, we can assume $q_n = q_{F_n} \to p$ uniformly on compact subsets of the open disk. $p$ may not be inner, but $p \in H^\infty$ and $| p(\lambda_i)| \geqslant 1/j$. For $H^\infty$ functions, bounded pointwise convergence implies weak * convergence in $L^\infty$ [3, p. 98] and therefore $q_n F_n \to pF$ weakly in $H^2{}_{\mathscr{K}}$. But $\mathcal{M}$ is weakly closed and therefore $pF \in \mathcal{M}$. If $r$ is the inner factor of $p$, it is easy to see $re \in \mathcal{M}$ and $| r(\lambda_i)| \geqslant 1/j$. Thus, $F \in \Gamma_{ij}$ and we apply the Baire category theorem to conclude that for some $m$, $n$, $\Gamma_{m,n}$ contains a ball $B = \{G : G \in K \text{ and } \| G - G_0 \| < \epsilon\}$. Let $F \in K$, $F \neq 0$. Then $G = G_0 + \epsilon F/(2 \| F \|) \in B$, $| q_G(\lambda_m)| \geqslant 1/n$ and $| q_{G_0}(\lambda_m)| \geqslant 1/n$. Clearly $q_G = q_{\alpha G}$ for complex $\alpha \neq 0$ and therefore for all $F \in K$, $| q_F(\lambda_m)| \geqslant 1/n^2$. Let $F_1, F_2, \ldots$ be an orthonormal basis for $K$ with minimal inner functions $q_1, q_2, \ldots$. To prove the theorem it suffices to show the $q_n$'s have a common inner multiple. By the lemma we can, for any $n$, find an $F \in K$ such that $q_F = q_1 \wedge \cdots \wedge q_n$. By the canonical infinite product representation, either $\{q_1 \wedge \cdots \wedge q_n\}$ converges uniformly on compact sets to an inner function or diverges to 0. The second possibility contradicts the fact that $\Gamma_{m,n^2} = K$, and the proof is finished.

Theorem 2 can be generalized as follows, as pointed out to author by Sz.-Nagy and Foiaş:

THEOREM 3. *Let $T$ be a contraction on $H$ and let $U_+$ acting on $H_+$ be its minimal isometric dilation* [3, p. 11]. *Let $K$ be a closed subspace of $H_+$ such that $\vee_0^\infty U_+^n K = H_+$, and let $P$ be the projection of $H_+$ onto $H$. Then, if for all $e \in PK$, the restriction of $T$ to the $T$ cyclic subspace of $H$ generated by $e$ belongs to $C_0$, then $T$ itself belongs to $C_0$.*

To get Theorem 2 from Theorem 3, let $T = PS \mid H$, where $H = H^2{}_{\mathscr{H}} \ominus \mathscr{M}$ and $P$ is the projection of $H^2{}_{\mathscr{H}}$ onto $H$, and observe that $S$ is the minimal isometric dilation of $T$. Theorem 3 would be equivalent to Theorem 2 if one could prove directly, under the hypothesis of Theorem 3, that $T^{*n} \to 0$ strongly. (See the remarks preceding Lemma 1.) We are unable to complete the proof along these lines, and Theorem 3 must apparently be proved by joining, with suitable modifications, the argument of Ref. [4] with our proof of Theorem 2. A natural conjecture arises from attempts to derive Theorem 3 from Theorem 2; viz., can one prove $T^{*n} \to 0$ strongly without using all available structure? The following statement, if true, is a local characterization of the class $C_{00}$ of contractions such that $T^n \to 0$ strongly and $T^{*n} \to 0$ strongly.

*Conjecture.* Let $T_e = T \mid \mathscr{L}(e)$, where $\mathscr{L}(e) = \vee_0^\infty T^n e$. Suppose for all $e \in \mathscr{H}$, $T_e^n \to 0$ strongly (which implies $T^n \to 0$ strongly) and $(T_e)^{*n} \to 0$ strongly. Does it follow that $T^{*n} \to 0$ strongly?

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